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# INVESTIGATION OF QUALITATIVE PROPERTIES OF THE NON-DIVERGENT CROSS-DIFFUSION PROBLEM WITH SOURCE AND VARIABLE DENSITY IN TWO-COMPONENT MEDIA IN CRITICAL CASES

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**ABSTRACT**

In this paper, we study the properties of self-similar solutions of a cross-diffusion parabolic system. The asymptotic behavior of self-similar solutions are analyzed for both the slow and fast diffusive regimes. It is shown that coefficients of the main term of the asymptotic of solution satisfy some system of nonlinear algebraic equations.

**Keywords:** cross-diffusive system, non-divergence form, global solutions.

## 1.Introduction

Consider in $Q=\{(t,x):\ t>0,\ x\in R^N\}$ the cross-diffusive system of equations in non-divergence form with Cauchy conditions:

$$|x|^{-l}\frac{\partial u_i}{\partial t}=u_i^{\alpha_i}\nabla\left(|x|^n u_{3-i}^{m_i-1}\left|\nabla u_i^k\right|^{p-2}\nabla u_i\right)+|x|^{-l}u_i^{\beta_i} \quad (1)$$

$$u_i(0,x)=u_{0,i}(x),\qquad x\in R^N \quad (i=1,2) \quad (2)$$

where $l,n,\alpha_i,k,\beta_i>0,\ p\geq 2,m_i\geq 1,(i=1,2)$ the numerical parameters, $\nabla(\cdot)=grad_x(\cdot)$, $u_i=u_i(x,t)\geq 0$ are the solutions. It is clear that the system (1) is degenerate. Therefore, it does not have classical solutions on the domain defined by equations. $u_i(t,x)=0$, $\nabla u_i(t,x)=0$ meaning system (1) may not have a classical solution. Therefore, in this case we consider a weak solution having the property $u_i^{\alpha_i}\nabla\left(|x|^n u_{3-i}^{m_i-1}\left|\nabla u_i^k\right|^{p-2}\nabla u_i\right)\in C(Q)\ (i=1,2)$. and obeying to the system (1) in sense of a distribution [1]. Non-divergent form equations and system of equations (1) are often used to describe various physical phenomena, such as the diffusive process for biological species, the resistive diffusion phenomena in force-free magnetic fields, curve shortening flow, spreading of infectious disease and so on, see for [1-18].

**2. The self-similar system of equations**

Here, we provide a method of nonlinear splitting [2] for construction of self-similar equation for the system given by Eqs. (1). We look for the solutions $u_i(t,x)(i=1,2)$ in the form

$$u_i(x,t)=\bar{u}_i(t)\cdot w_i(\tau(t),\varphi(|x|))\,(i=1,2) \qquad (3)$$

Then, we obtain $\bar{u}_i(t)$ as $\bar{u}_i(t)=(T+t)^{\frac{1}{1-\beta_i}}\,(i=1,2)$.

**Case 1.** Let $p>l+n$ and $\frac{k(p-2)+\alpha_1}{1-\beta_1}+\frac{m_1-1}{1-\beta_2}+1>0$ be.From Eqs. (3) and (1), we obtain the following system of equations:

$$\frac{\partial w_i}{\partial\tau}=w_i^{\alpha_i}\varphi^{1-s}\frac{\partial}{\partial\varphi}\left(\varphi^{s-1}w_{3-i}^{m_i-1}\left|\frac{\partial w_i^k}{\partial\varphi}\right|^{p-2}\frac{\partial w_i}{\partial\varphi}\right)+\frac{\psi_i}{\tau}\left(w_i^{\beta_i}-\frac{1}{1-\beta_i}w_i\right)(i=1,2) \qquad (4)$$

where:

$$\tau(t)=\begin{cases}\dfrac{(T+t)^{\sigma}}{\sigma} & at \quad \sigma\neq 0\\ \ln(T+t) & at \quad \sigma=0\end{cases}$$

$$T>0,\sigma=\frac{k(p-2)+\alpha_1}{1-\beta_1}+\frac{m_1-1}{1-\beta_2}+1,\quad \psi_i=\left(\frac{k(p-2)+\alpha_i}{1-\beta_i}+\frac{m_i-1}{1-\beta_{3-i}}+1\right)^{-1},\varphi(|r|)=r^{\frac{p-n-l}{p}},$$

$$s=\frac{p(N-1)+n+l-p(l-1)}{p},\quad r=\sqrt{\sum_{i=1}^{N}x_i^2}$$

$$\frac{k(p-2)+\alpha_1}{1-\beta_1}+\frac{m_1-1}{1-\beta_2}=\frac{k(p-2)+\alpha_2}{1-\beta_2}+\frac{m_2-1}{1-\beta_1}$$

It is easy to establish that the system (4) has the self-similar solution

$$w_1(x,\tau)=f_1(\xi),w_2(x,\tau)=f_2(\xi),\ \xi=\frac{\varphi(|r|)}{\tau^{\frac{1}{p}}} \qquad (5)$$

where ξ is self-similar variable and the functions $f_1(\xi), f_2(\xi)$ fulfill the following approximately self-similar system of equations

$$f_i^{\alpha_i}\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f_{3-i}^{m_i-1}\left|\frac{df_i^k}{d\xi}\right|^{p-2}\frac{df_i}{d\xi}\right)+\frac{\xi}{p}\frac{df_i}{d\xi}+\psi_i\left(f_i^{\beta_i}-\frac{1}{1-\beta_i}f_i\right)=0 \qquad (6)$$

In the following, we will consider nontrivial, nonnegative solutions of the system (6) satisfying the following conditions:

$$\begin{aligned} &f_1(0)=M_1,\ f_2(0)=M_2, M_1\in R, M_2\in R \\ &f_1(d_1)=f_2(d_2)=0,\ 0<d_1<\infty,\ 0<d_2<\infty \end{aligned} \tag{7}$$

## 3. EXPLICIT ESTIMATE AND GLOBAL SOLUTION

Using the solution comparison method of [1] and the standard equations method of [2] for solving the problem (6)-(7), we can obtain the estimates for the solution of the problem (1)-(2).

We note that the functions:

$$f_1(\xi)=A_1\left(a-\xi^{\gamma}\right)^{\gamma_1},\ f_2(\xi)=A_2\left(a-\xi^{\gamma}\right)^{\gamma_2} \tag{8}$$

$$\gamma=\frac{p}{p-1},\ \gamma_i=\frac{(p-1)\left(k(p-2)+\alpha_{3-i}-m_i+1\right)}{k(p-2)\left(k(p-2)+\alpha_i+\alpha_{3-i}\right)+\alpha_i\alpha_{3-i}-(m_i-1)(m_{3-i}-1)}$$

$$b_i=(\gamma_i k-1)(p-2)+\gamma_{3-i}(m_i-1)+\gamma_i-1$$

$A_i\ (i=1,2)$ found numbers. $(b)_+=max(0,b)$

The following theorem is proved.

**Theorem** 1. Let the conditions of $\gamma_i>0$

$$\gamma\gamma_{i+2}A_i^{\alpha_i+k(p-2)}\cdot A_{3-i}^{m_i-1}k^{p-2}\left|\gamma\gamma_i\right|^{p-2}=\frac{1}{p}$$

$$\psi_i\left(A_i^{\beta_i-1}a^{\gamma_i\beta_i-\gamma_i}-\frac{1}{1-\beta_i}\right)-\frac{s\gamma_i}{p\gamma_{i+2}}\le 0$$

$$u_i(t,0)\le u_{+i}(t,0),\ x\in R^N\ i=1,2$$

Then the problem (1)-(2) has a global solution, which satisfies estimates

$$u_i(t,x)\le u_{i+}(t,x)=(T+t)^{\frac{1}{1-\beta_i}}\cdot A_i\left(a-\xi^{\gamma}\right)_+^{\gamma_i} \quad \text{in Q.}$$

**Proof.** Theorem 1 is proved by the comparing solution method [1]. Hence, comparing solution methods it is taken the functions $u_{i+}(t,x)$. Substituting (8) in (1) the following inequality can be obtained: $f_i^{\alpha_i}\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f_{3-i}^{m_i-1}\left|\frac{df_i^k}{d\xi}\right|^{p-2}\frac{df_i}{d\xi}\right)+\frac{\xi}{p}\frac{df_i}{d\xi}+\psi_i\left(f_i^{\beta_i}-\frac{1}{1-\beta_i}f_i\right)\le 0$

(9)

If the specific form (8) is given for the functions $f_i(\xi)(i=1,2)$ inequality (9) can be rewritten as follows:

$$\psi_i\left(A_i^{\beta_i-1}\left(a-\xi^{\gamma}\right)^{\gamma_i\beta_i-\gamma_i}-\frac{1}{1-\beta_i}\right)-\frac{\gamma_i s}{\gamma_{i+2}p}\le 0$$

It is easy to check that $A_i^{\beta_i-1}a^{\gamma_i\beta_i-\gamma_i}\ge A_i^{\beta_i-1}\left(a-\xi^{\gamma}\right)^{\gamma_i\beta_i-\gamma_i}$

Then, according to the hypotheses of Theorem 1 and comparison principle, it will be: $u_i(t,x)\le u_{i+}(t,x)$ , $x\in R^N$ in $Q$ ,

if $u_i(t,x)\le u_{i+}(t,x)$ , $x\in R^N$.

The results obtained in the above theorem are a generalization of the results in [18]. If we take $k=1$ and $p=2$, in our given theorem, then the results in [18] are obtained.

## 4. The asymptotic behavior of self-similar solutions of the problem (6)-(7)

### 4.1. The case of slow diffusion.

Let us introduce the following notations:

$$a_{1i}(\eta)=-\gamma_3+\frac{se^{-\eta}}{\gamma\left(a-e^{-\eta}\right)},\ a_{2i}(\eta)=\frac{1}{p\gamma^{p-1}},\ a_{3i}(\eta)=\frac{\psi_i e^{-(1+\gamma_1\beta-\gamma_1)\eta}}{\gamma^p\left(a-e^{-\eta}\right)},\ a_{4i}(\eta)=\frac{\psi_i e^{-\eta}}{\gamma^p\left(1-\beta_1\right)\left(a-e^{-\eta}\right)}$$

Assume $\frac{k(p-2)+\alpha_1}{1-\beta_1}+\frac{m_1-1}{1-\beta_2}=\frac{k(p-2)+\alpha_2}{1-\beta_2}+\frac{m_2-1}{1-\beta_1}$.Then the following theorem is valid:

**Theorem 2.** Let $\gamma_i>0$. Then compactly supported solution of the problem (6),(7) as $|x|\to a^{\frac{p-1}{p}}\tau^{\frac{1}{p}}$ has the following asymptotic behavior:

$$f(\xi)=c_i\left(a-\left(\frac{|x|}{\tau^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}\right)^{\gamma_i}\left(1+o(1)\right) \qquad (10)$$

if one of the the following conditions are fulfilled:

(1) $1+\gamma_1\beta_1-\gamma_1=0$ and $1+\gamma_2\beta_2-\gamma_2=0$ the coefficients $c_i\ (i=1,2)$ are the solutions of the systems of nonlinear algebraic equations:

$$a_{11}c_1^{k(p-2)+1}c_2^{m_1-1}k^{p-2}\gamma_1^{p-1}+a_{12}c_1^{1-\alpha_1}\gamma_1-a_{13}c_1^{\beta_1-\alpha_1}=0$$
$$a_{21}c_2^{k(p-2)+1}c_1^{m_2-1}k^{p-2}\gamma_2^{p-1}+a_{22}c_2^{1-\alpha_2}\gamma_2-a_{23}c_2^{\beta_2-\alpha_2}=0$$

(2) $1+\gamma_1\beta_1-\gamma_1=0$ and $1+\gamma_2\beta_2-\gamma_2>0$ the coefficients $c_i\ (i=1,2)$ are the solutions of the systems of nonlinear algebraic equations:

$$a_{11}c_1^{k(p-2)+1}c_2^{m_1-1}k^{p-2}\gamma_1^{p-1}+a_{12}c_1^{1-\alpha_1}\gamma_1-a_{13}c_1^{\beta_1-\alpha_1}=0$$
$$a_{21}c_2^{k(p-2)+1}c_1^{m_2-1}k^{p-2}\gamma_2^{p-1}+a_{22}c_2^{1-\alpha_2}\gamma_2=0$$

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(3) $1+\gamma_1\beta_1-\gamma_1>0$ and $1+\gamma_2\beta_2-\gamma_2=0$ the coefficients $c_i\,(i=1,2)$ are the solutions of the systems of nonlinear algebraic equations:

$$a_{11}c_1^{k(p-2)+1}c_2^{m_1-1}k^{p-2}\gamma_1^{p-1}+a_{12}c_1^{1-\alpha_1}\gamma_1=0$$

$$a_{21}c_2^{k(p-2)+1}c_1^{m_2-1}k^{p-2}\gamma_2^{p-1}+a_{22}c_2^{1-\alpha_2}\gamma_2-a_{23}c_2^{\beta_2-\alpha_2}=0$$

(4) $1+\gamma_1\beta_1-\gamma_1>0$ and $1+\gamma_2\beta_2-\gamma_2>0$ the coefficients $c_i\,(i=1,2)$ are the solutions of the systems of nonlinear algebraic equations:

$$a_{11}c_1^{k(p-2)+1}c_2^{m_1-1}k^{p-2}\gamma_1^{p-1}+a_{12}c_1^{1-\alpha_1}\gamma_1=0$$

$$a_{21}c_2^{k(p-2)+1}c_1^{m_2-1}k^{p-2}\gamma_2^{p-1}+a_{22}c_2^{1-\alpha_2}\gamma_2=0$$

**Proof.** The proof of this theorem is given in [4].

**Corollary 1**. If inequality $\gamma_i>0$ holds, then the generalized solution of problem (3.2.1)-(3.2.2) has the following

$$u_{iA}(x,t)\approx c_i(T+t)^{\frac{1}{1-\beta_i}}\left(a-\left(|x|\tau^{-\frac{1}{p}}\right)^{\frac{p}{p-1}}\right)^{\gamma_i}(1+o(1))$$

asymptotics in $|x|\to a^{\frac{p-1}{p}}\tau^{\frac{1}{p}}$, where $c_i\,(i=1,2)$ are definite constants.

**Case 2.** Let $p=l+n$ and $\frac{k(p-2)+\alpha_1}{1-\beta_1}+\frac{m_1-1}{1-\beta_2}+1>0$ be. From Eqs.(3) and (1), we obtain the following system of equations:

$$\frac{\partial w_i}{\partial\tau}=w_i^{\alpha_i}\frac{\partial}{\partial\varphi}\left(w_{3-i}^{m_i-1}\left|\frac{\partial w_i^k}{\partial\varphi}\right|^{p-2}\frac{\partial w_i}{\partial\varphi}\right)+w_i^{\alpha_i}\left(w_{3-i}^{m_i-1}\left|\frac{\partial w_i^k}{\partial\varphi}\right|^{p-2}\frac{\partial w_i}{\partial\varphi}\right)+\quad (i=1,2)\qquad (11)$$

$$\frac{\psi_i}{\tau}\left(w_i^{\beta_i}-\frac{1}{1-\beta_i}w_i\right),(i=1,2)$$

where:

$$\tau(t)=\frac{(T+t)^{\sigma}}{\sigma},\varphi(|x|)=\ln|x|,T>0\ \psi_i=\left(\frac{k(p-2)+\alpha_i}{1-\beta_i}+\frac{m_i-1}{1-\beta_{3-i}}+1\right)^{-1}(i=1,2)$$

$$\sigma=\frac{k(p-2)+\alpha_1}{1-\beta_1}+\frac{m_1-1}{1-\beta_2}+1=\frac{k(p-2)+\alpha_2}{1-\beta_2}+\frac{m_2-1}{1-\beta_1}+1$$

Given that this system of equations represents a physical process, the solution can be estimated from above, knowing that the continuous flow is

$$w_i^{\alpha_i}\left(w_{3-i}^{m_i-1}\left|\frac{\partial w_i^k}{\partial\varphi}\right|^{p-2}\frac{\partial w_i}{\partial\varphi}\right)\le 0.$$

**Theorem 3.** Let the conditions of $\gamma_i > 0$

$$\gamma\gamma_{i+2} A_i^{\alpha_i + k(p-2)} \cdot A_{3-i}^{m_i - 1} k^{p-2} \left|\gamma\gamma_i\right|^{p-2} = \frac{1}{p}$$

$$\frac{1}{p(1-\alpha_i)} + \psi_i \left( A_i^{\beta_i - 1} a^{\gamma_i \beta_i - \gamma_i} - \frac{1}{1-\beta_i} \right) \le 0$$

$$u_i(t,0) \le u_i(t,0)_+, x \in R^N \; i = 1,2$$

Then the problem (1)-(2) has a global solution, which satisfies estimates

$$u_i(t,x) \le u_{i+}(t,x) = (T+t)^{\frac{1}{1-\beta_i}} \cdot A_i \left(a - \xi^\gamma\right)_+^{\gamma_i} \qquad \text{in } Q \setminus \{0\}.$$

Let $p = l + n$ and $\frac{k(p-2)+\alpha_1}{1-\beta_1} + \frac{m_1 - 1}{1-\beta_2} + 1 > 0$ be.

**Case 3.** Let $p = l + n$ and $\frac{k(p-2)+\alpha_1}{1-\beta_1} + \frac{m_1 - 1}{1-\beta_2} = 0$ be. As mentioned in case 2, we transfer (3) to the system of equations (1). Here functions $\tau(t)$, $\varphi(|x|)$ are defined in $\tau(t) = (T+t), \varphi(|x|) = \ln|x|, T > 0$ ways. If we transfer functions (3) to the system of equations (1), as a result we will have the system of equations (11). Theorem 3 is applicable here.

**Case 4.** Let $p = l + n$ and $\beta_i = 1\ (i = 1,2)$ be. We will look for the solution of the problem (1)-( 2) in the following form:

$$u_i(x,t) = \bar{u}_i(t) \cdot w_i(\tau(t), \varphi(|x|))\ (i = 1,2)$$

Given that $\bar{u}_i(t) = e^t\ (i = 1,2)$ and $\tau(t) = \frac{e^{(\alpha_1 + m_1 + k(p-2))t}}{(\alpha_1 + m_1 + k(p-2))}$ are here.

We will have the following system of equations.

$$\frac{\partial w_i}{\partial \tau} = w_i^{\alpha_i} \frac{\partial}{\partial \varphi} \left( w_{3-i}^{m_i - 1} \left| \frac{\partial w_i^k}{\partial \varphi} \right|^{p-2} \frac{\partial w_i}{\partial \varphi} \right) (i = 1,2) \qquad (12)$$

If we make $w_i(\tau(t), \varphi(|x|)) = f_i(\xi)$ $,(i = 1,2)$ substitutions in (12), then we will have the following system of automodel equations.

$$f_i^{\alpha_i} \frac{d}{d\xi} \left( f_{3-i}^{m_i - 1} \left| \frac{df_i^k}{d\xi} \right|^{p-2} \frac{df_i}{d\xi} \right) + \frac{\xi}{p} \frac{df_i}{d\xi} = 0\ (i = 1,2) \qquad (13)$$

We deal with finding non-negative solutions of the system of equations (13) satisfying the conditions (7).

**Slow diffusion.** In order to derive the global conditions of the solution in the case of slow diffusion, we look for solutions of $u_i(t,x)_+ \ (i=1,2)$ in the following form.

$$u_i(t,x)_+ = e^t \bar{f}_i(\xi),\ \bar{f}_i(\xi) = A_i\left(a-\xi^{\gamma}\right)_+^{\gamma_i}$$

**Theorem 4.** Let the conditions of $\gamma_i > 0$, $p(1-\alpha_i) \le 0$

$$\gamma\gamma_{i+2} A_i^{\alpha_i + k(p-2)} \cdot A_{3-i}^{m_i-1} k^{p-2} \left|\gamma\gamma_i\right|^{p-2} = \frac{1}{p}$$

$$u_i(t,0) \le u_i(t,0)_+,\ x \in R^N\ i=1,2$$

Then the problem (1)-(2) has a global solution, which satisfies estimates

$$u_i(t,x) \le u_{i+}(t,x) \qquad \text{in } Q \setminus \{0\}.$$

**Fast diffusion.** Suppose that the system (13) satisfies the conditions

$$f_i'(0)=0,\ f_i(\infty)=0.$$

**Theorem 5.** Let the conditions of $\gamma_i < 0$, $p(1-\alpha_i) \ge 0$

$$\gamma\gamma_{i+2} A_i^{\alpha_i + k(p-2)} \cdot A_{3-i}^{m_i-1} k^{p-2} \left|\gamma\gamma_i\right|^{p-2} = -\frac{1}{p}$$

$$u(t,0) \ge u(t,0)_-,\ x \in R^N\ i=1,2$$

Then the problem (1)-(2) has a global solution, which satisfies estimates

$$u_i(t,x) \le u_{i+}(t,x) \qquad \text{in } Q \setminus \{0\}.$$

**Computational experiment and visualization results.**

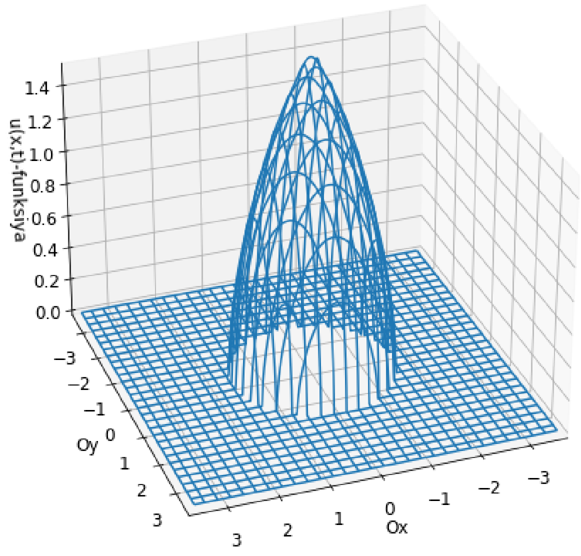
We solve the problem numerically for the case where $Q=\{(t,x): t>0, x\in R^N\}$ is $N=2$ in the domain (1)-(2).

$$\begin{cases} |x|^n \frac{\partial u}{\partial t} = u^{\alpha_1}\frac{\partial}{\partial x_1}\left(v^{m_1-1}\left|\frac{\partial u^k}{\partial x_1}\right|^{p-2}\frac{\partial u}{\partial x_1}\right) + u^{\alpha_1}\frac{\partial}{\partial x_2}\left(v^{m_1-1}\left|\frac{\partial u^k}{\partial x_1}\right|^{p-2}\frac{\partial u}{\partial x_1}\right) + |x|^n u^{\beta_1} \\ |x|^n \frac{\partial v}{\partial t} = v^{\alpha_1}\frac{\partial}{\partial x_1}\left(u^{m_1-1}\left|\frac{\partial v^k}{\partial x_1}\right|^{p-2}\frac{\partial v}{\partial x_1}\right) + v^{\alpha_1}\frac{\partial}{\partial x_2}\left(u^{m_1-1}\left|\frac{\partial v^k}{\partial x_1}\right|^{p-2}\frac{\partial v}{\partial x_1}\right) + |x|^n v^{\beta_2} \end{cases}$$

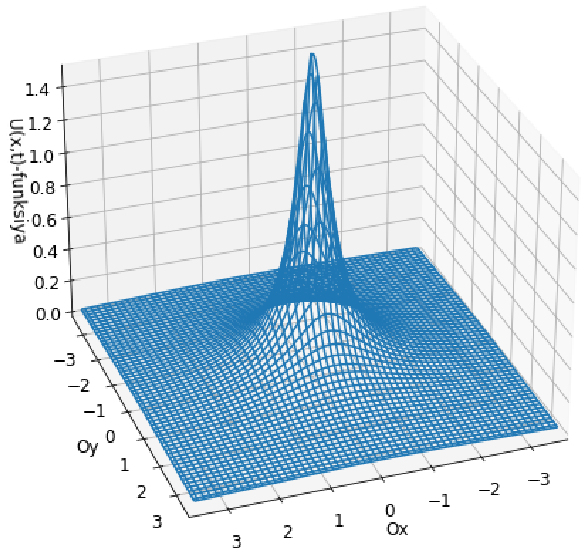
$$u(0,x)=u_0(x)\ge 0, v(0,x)=v_0(x)\ge 0,\ x\in R^2$$

Here it is defined as $u=u(t,x_1,x_2), v=v(t,x_1,x_2)$, $|x|=\sqrt{x_1^2+x_2^2}$.

In order to numerically solve this problem, we use one of the variable direction methods, the "longitudinal-transverse" differential scheme, in other words, the Pisman-Reichford differential scheme.




$$k=1.2, p=6.8, m_1=1.35, m_2=1.45$$
$$\alpha_1=0.2, \alpha_2=0.2, \beta_1=0.4, \beta_2=0.6$$




$$k=1.2, p=2.8, m_1=1.37, m_2=1.46$$
$$\alpha_1=0.2, \alpha_2=0.2, \beta_1=0.4, \beta_2=0.6$$

where Figure 1 represents the slow diffusion process and Figure 2 represents the fast diffusion process.

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